Lecture 17: Approximate Integration 7.7

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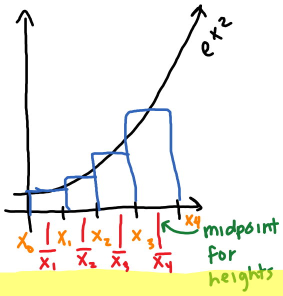
Example

$$\int_0^1 e^{x^2}\,dx$$

→ we can't solve this with the methods we know

Definite integral: approximate area under the curve
→ can use Riemann sums to approximate



Midpoint Rule

$$\int_a^b f(x)\,dx \approx M_n = \Delta x[f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_n})]$$

where $\Delta x = \frac{b-a}{n}, \overline{x_i} = \frac{1}{2}(x_i + x_{i+1})$
(midpoint of $[x_i, x_{i+1}]$)

(→ same as twice as amny rectangles with left and right hand points mixed)

$$\int_0^1 e^{x^2}\,dx$$

→ break up into 4 rectangles with a midpoint as height (see picture)

First, find interval $[x_i, x_{i+1}]$.

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}, \qquad \text{then each } x_i = x_0 + \Delta x \cdot i$$

$$= 0 + \frac{1}{4} \cdot i = \frac{i}{4}$$

then, compute $\overline{x_i}$:

$$\overline{x_i} = \frac{1}{2}(x_i + x_{i+1}) = \frac{1}{2}\left(\frac{i}{4} + \frac{i+1}{4}\right) = \frac{i+i+1}{4} = \frac{2i+1}{4} = \frac{i}{4} + \frac{1}{8}$$

$$\overline{x_i} = \left\{\frac{1}{8}, \frac{1}{4} + \frac{1}{8}, \frac{2}{4} + \frac{1}{8}, \frac{3}{4} + \frac{1}{8}\right\} = \boxed{\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}}$$

↳ our midpoints

$$M_4 = \frac{1}{4} \cdot \left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right]$$

Δx ←

$$= \frac{1}{4}\left(e^{\left(\frac{1}{8}\right)^2} + e^{\left(\frac{3}{8}\right)^2} + e^{\left(\frac{5}{8}\right)^2} + e^{\left(\frac{7}{8}\right)^2}\right)$$

We want to estimate the error we made without knowing the exact value of the integral $\int_0^1 e^{x^2}\,dx$ (we can't compute it):

ERROR for $\int^b f(x)\,dx$:

integral $\int_0 e^{x^2}\ dx$ (we can't compute it):

ERROR for $\int_a^b f(x)\ dx$:

$$|E_M| \le K \cdot \frac{(b-a)^2}{24 \cdot n^2}$$

where K is such that $|f''(x)| \le K$ for all $a \le x \le b$

<u>for the example</u>:

we had $n = 4, f(x) = e^{x^2}$

$f'(x) = e^{x^2} \cdot 2x$

$f''(x) = \left(e^{x^2} \cdot 2x\right) \cdot 2x + e^{x^2} \cdot 2$

$\qquad = 4x^2 e^{x^2} + 2e^{x^2}$

<u>we need to find a bound such that</u> $f''(x) \le K$ on $[0,1]$

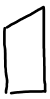
$f''(x)$ is strictly increasing on [0,1], so max is at x=1 and min is at x=0.

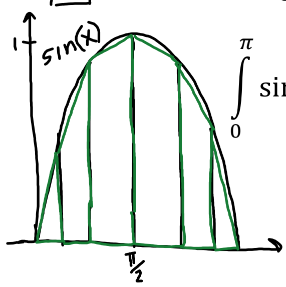
$\underline{f''(1)} = 4 \cdot 1^2 \cdot e^{1^2} + 2 \cdot e^{1^2} = \underline{6e}$

so $|E_M| = \frac{\cancel{6}\cdot e(1-0)^2}{4\,\cancel{24}\cdot 4^2} = \frac{e}{4^3} = \frac{e}{2^6} = \frac{e}{64}$

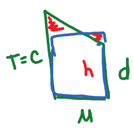
<u>Better approximation method I</u>:

<u>Trapezoid Rule</u>:

use  instead of 



$$\int_0^\pi \sin(x)$$



area of T:

$$A = \frac{1}{2}\underbrace{(c+d)}_{h} \cdot u$$

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\big(\overbrace{f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_i)}^{n+1 \text{ terms}}\big)$$

$$\Delta x = \frac{b-a}{n}, \qquad x_i = a + i \cdot \Delta x \ \text{(as in Riemann sum)}$$

$$\int_0^\pi \sin(x)\,dx, \qquad \text{compute } T_6, \Delta x = \frac{\text{pi} - 0}{6} = \frac{pi}{6}$$

$$T_6 = \frac{\Delta x}{2}\left(f(0) + 2f\left(\frac{pi}{6}\right) + 2f\left(\frac{pi}{3}\right) + 2f\left(\frac{pi}{2}\right) + 2f\left(\frac{2pi}{3}\right) + 2f\left(\frac{5pi}{6}\right) + f(pi)\right.$$

$$= \frac{pi}{6 \cdot 2}\left(\sin(0) + 2\sin\left(\frac{pi}{6}\right) + 2\sin\left(\frac{pi}{3}\right) + 2\sin\left(\frac{pi}{2}\right) + 2\sin\left(\frac{2pi}{3}\right) + 2\sin\left(\frac{5pi}{6}\right)\right.$$

$$+ \sin(pi)$$

$$= \frac{pi}{12}\left(0 + \cancel{2}\cdot\frac{1}{\cancel{2}} + \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}} + 2\cdot 1 + \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}} + \cancel{2}\cdot\frac{1}{\cancel{2}} + 0\right)$$

$$= \frac{pi}{12}\left(1 + \sqrt{3} + 2 + \sqrt{3} + 1\right)$$

$$= \frac{pi}{12}\left(4 + 2\sqrt{3}\right) \text{ factor out 2}$$

$= \frac{pi}{6}(2+\sqrt{3})$

ERROR

If $f''(x) \le K$ on $a \le x \le b$ then:

$|E_T| \le \frac{K \cdot (b-a)^3}{12 \cdot n^2}$

Example: $n = 6, f'(x) = \cos(x), f''(x) = -\sin(x)$

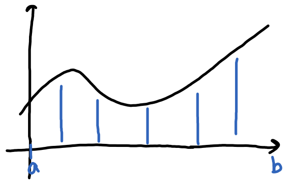
so: $|f''(x)| \le 1$ for all $x \in [0, \pi]$

so: $k = 1$ is a good choice.

$|E_T| = \frac{1 \cdot (\pi - 0)^3}{12 \cdot 6^2} = \frac{\pi^3}{442} \approx 0.071499 \ldots$

Simpson's Rule: better than T_n

idea: approximate graph by parabolas



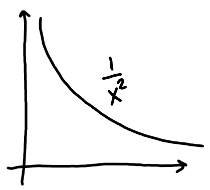
$$\int_a^b f(x)\,dx = \frac{\Delta x}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

$$\Delta x = \frac{b-a}{n}$$

$*$ n has to be even for this to work! $*$

Example

$\int_1^2 \frac{1}{x^2}\,dx$ 8 steps



$\Delta x = \frac{2-1}{8} = \frac{1}{8}$

S_8

$$= \frac{1}{8 \cdot 3}\left(f(1) + 4f\left(\frac{1}{8}\right) + 2f\left(\frac{5}{4}\right) + 4f\left(1 + \frac{3}{8}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(1 + \frac{5}{8}\right)\right.$$

$$\left. + 2f\left(1 + \frac{3}{4}\right) + \cdots + f(2)\right)$$

$$= \frac{1}{24}\left(\frac{1}{1^2} + \frac{4(1)}{\left(\frac{9}{8}\right)^2} + \cdots + \frac{2 \cdot 1}{\left(\frac{15}{8}\right)^2} + \frac{1}{2^2}\right)$$

$$= \frac{1}{24}\left(1 + \frac{4 \cdot 8^2}{9^2} + \cdots + \frac{2 \cdot 8^2}{15^2} + \frac{1}{4}\right)$$

ERROR

if $|f^4(x)| \le K$ then:

$|E_S| \le \frac{k \cdot (b-a)^5}{180 \cdot n^4}$

In example: $f^4(x) = \cdots$

$f(x) = x^{-2}, \quad f'(x) = -2x^{-3}, \quad f''(x) = (-2) \cdot (-3)x^{-4} = 6x^{-5}$

$$f'''(x) = 6 \cdot (-4) \cdot x^{-5}, \qquad f^4(x) = (-24)(-5) \cdot x^{-6} = \frac{120}{x^6}$$

$f^{(4)}(1) = 120,$ $\qquad f^4(2) = \frac{120}{2^6}$

value for k

$$|E_S| \leq \frac{\overset{2}{\cancel{120}}(2-1)^5}{\underset{3}{\cancel{180}} \cdot 8^4} = \frac{2}{3 \cdot (2^3)^4} = \frac{2}{3 \cdot 2^{12}} = \frac{1}{3 \cdot 2^{11}}$$